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KOSZUL HOMOLOGY OF LOCAL COHOMOLOGY MODULES

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Abstract. Let K be a field of characteristic zero, $R = K[X_1, \ldots, X_n]$ and let I be an ideal in R. Let $A_n(K) = K < X_1, \ldots, X_n, \partial_1, \ldots, \partial_n >$ be the n^{th} Weyl algebra over K. By a result due to Lyubeznik the local cohomology modules $H^i_I(R)$ are holonomic $A_n(K)$-modules for each $i \geq 0$. In this article we compute the Koszul homology modules $H_*(\partial_1, \ldots, \partial_n; H^*_I(R))$ for certain classes of ideals.

Introduction

Let K be a field of characteristic zero, $R = K[X_1, \ldots, X_n]$ and let I be an ideal in R. For $i \geq 0$ let $H^i_I(R)$ be the i^{th}-local cohomology module of R with respect to I. Let $A_n(K) = K < X_1, \ldots, X_n, \partial_1, \ldots, \partial_n >$ be the n^{th} Weyl algebra over K. By a result due to Lyubeznik, see [5], the local cohomology modules $H^i_I(R)$ are finitely generated $A_n(K)$-modules for each $i \geq 0$. In fact they are *holonomic* $A_n(K)$ modules. In [1] holonomic $A_n(K)$ modules are denoted as $\mathcal{B}_n(K)$, the *Bernstein* class of left $A_n(K)$ modules.

Let N be a left $A_n(K)$ module. Now $\partial = \partial_1, \ldots, \partial_n$ are pairwise commuting K-linear maps. So we can consider the Koszul complex $K(\partial; N)$. Notice that the homology modules $H_*(\partial; N)$ are in general only K-vector spaces. They are finite dimensional if $N \in \mathcal{B}_n(K)$; [1, Chapter 1, Theorem 6.1]. In particular $H_*(\partial; H^*_I(R))$ are finite dimensional K-vector spaces. In this paper we compute it for a few classes of ideals.

Throughout let $K \subseteq L$ where L is an algebraically closed field. Let $A^n(L)$ be the affine n-space over L. If I is an ideal in R then

$$V(I)_L = \{\mathbf{a} \in A^n(L) \mid f(\mathbf{a}) = 0; \text{ for all } f \in I\};$$

denotes the variety of I in $A^n(L)$. By Hilbert's Nullstellensatz $V(I)_L$ is always non-empty. We say that an ideal I in R is zero-dimensional if $\ell(R/I)$ is finite and non-zero (here $\ell(-)$ denotes length). This is equivalent to saying that $V(I)_L$ is a finite non-empty set. If S is a finite set then let $\sharp S$ denote the number of elements in S. Our first result is

Theorem 1. *Let $I \subset R$ be a zero-dimensional ideal. Then $H_i(\partial; H^n_I(R)) = 0$ for $i \geq 1$ and*

$$\dim_K H_0(\partial; H^n_I(R)) = \sharp V(I)_L$$

For (irreducible) curves in $A^n(L)$ we have the following vanishing result:

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Theorem 2. *Let P be a height $n-1$ prime ideal in R. Then*

$$H_i\left(\partial; H^{n-1}_P(R)\right) = 0 \quad \textit{for } i \geq 2.$$

For homogeneous ideals it is best to consider their vanishing set in a projective case. Throughout let $P^{n-1}(L)$ be the projective $n-1$ space over L. We assume $n \geq 2$. Let I be a homogeneous ideal in R. Let

$$V^*(I)_L = \{\mathbf{a} \in P^{n-1}(L) \mid f(\mathbf{a}) = 0; \text{ for all } f \in I\};$$

denote the variety of I in $P^{n-1}(L)$. Note that $V^*(I)_L$ is a non-empty finite set if and only if $\operatorname{ht}(I) = n-1$. We prove

Theorem 3. *Let $I \subset R$ be a height $n-1$ homogeneous ideal. Then*

$$\begin{aligned}
\dim_K H_0(\partial; H^{n-1}_I(R)) &= \sharp V^*(I)_L - 1,\\
\dim_K H_1(\partial; H^{n-1}_I(R)) &= \sharp V^*(I)_L,\\
H_i(\partial; H^{n-1}_I(R)) &= 0 \textit{ for } i \geq 2.
\end{aligned}$$

For (irreducible) curves in $P^{n-1}(L)$ we have the following vanishing result:

Theorem 4. *Let P be a height $n-2$ homogeneous prime ideal in R. Then*

$$H_i\left(\partial; H^{n-2}_P(R)\right) = 0 \quad \textit{for } i \geq 3.$$

Note that for any non-zero ideal $H^0_I(R) = 0$. We prove the following vanishing result

Theorem 5. *Let I be a non-zero ideal in R. Then $H_n(\partial; H^1_I(R)) = 0$.*

Let M be a holonomic $A_n(K)$-module. By a result of Lyubeznik the set of associate primes of M as a R-module is finite. Note that the set $\operatorname{Ass}_R(M)$ has a natural partial order given by inclusion. We say P is a *maximal* isolated associate prime of M if P is a maximal ideal of R and also a minimal prime of M. We set $\operatorname{mIso}_R(M)$ to be the set of all maximal isolated associate primes of M. We show

Theorem 6. *Let M be a holonomic $A_n(K)$-module. Then*

$$\dim_K H_0(\partial; M) \geq \sharp \operatorname{mIso}_R(M).$$

We give an application of Theorem 6. Let I be an unmixed ideal of height $\leq n-2$. By Grothendieck vanishing theorem and the Hartshorne-Lichtenbaum vanishing theorem it follows that $H^{n-1}_I(R)$ is supported only at maximal ideals of R. By Theorem 6 we get

$$\sharp \operatorname{Ass}_R H^{n-1}_I(R) \leq \dim_K H_0\left(\partial; H^{n-1}_I(R)\right).$$

We now describe in brief the contents of the paper. In section 1 we discuss a few preliminary results that we need. In section 2 we make a few computations. This is used in section 3 to prove Theorem 1. In section 4 we make some additional computations and use it in section 5 to prove Theorem 3. In section 6 we prove Theorem 5. In section 7 we prove Theorem 6. In section 8 we prove Theorem 2. In section 9 we prove Theorem 4.

1. Preliminaries

In this section we discuss a few preliminary results that we need.

1.1. Let M be a holonomic $A_n(K)$-module. Then for $i = 0, 1$ the Koszul homology modules $H_i(\partial_n, M)$ are holonomic $A_{n-1}(K)$-modules, see [1, 1.6.2].

The following result is well-known.

Lemma 1.2. *Let $\partial = \partial_r, \partial_{r+1}, \dots, \partial_n$ and $\partial' = \partial_{r+1}, \dots, \partial_n$. Let M be a left $A_n(K)$-module. For each $i \geq 0$ there exist an exact sequence*

$$0 \to H_0(\partial_r; H_i(\partial'; M)) \to H_i(\partial; M) \to H_1(\partial_r; H_{i-1}(\partial'; M)) \to 0.$$

1.3. (linear change of variables). We consider a linear change of variables. Let $U_1, \dots, U_n$ be new variables defined by

$$U_i = d_{i1}X_1 + \cdots + d_{in}X_n + c_i \quad \text{for } i = 1, \dots, n$$

where $d_{ij}, c_1, \dots, c_n \in K$ are arbitrary and $D = [d_{ij}]$ is an invertible matrix. We say that the change of variables is homogeneous if $c_i = 0$ for all i.

Let $F = [f_{ij}] = (D^{-1})^{tr}$. Using the chain rule it can be easily shown that

$$\frac{\partial}{\partial U_i} = f_{i1}\frac{\partial}{\partial X_1} + \cdots + f_{in}\frac{\partial}{\partial X_n} \quad \text{for } i = 1, \dots, n.$$

In particular we have that for any $A_n(K)$ module M an isomorphism of Koszul homologies

$$H_i\left(\frac{\partial}{\partial U_1}, \cdots, \frac{\partial}{\partial U_n}; M\right) \cong H_i\left(\frac{\partial}{\partial X_1}, \cdots, \frac{\partial}{\partial X_n}; M\right)$$

for all $i \geq 0$.

1.4. Let I, J be two ideals in R with $J \supset I$ and let M be a R-module. The inclusion $\Gamma_J(-) \subset \Gamma_I(-)$ induces, for each i, an R-module homomorphism

$$\theta^i_{J,I}(M)\colon H^i_J(M) \to H^i_I(M).$$

If $L \supset J$ then we can easily see that

$$(\dagger) \qquad \theta^i_{J,I}(M) \circ \theta^i_{L,J}(M) = \theta^i_{L,I}(M).$$

Lemma 1.5. *(with hypotheses as above) If M is a $A_n(K)$-module then the natural map $\theta^i_{J,I}(M)$ is $A_n(K)$-linear.*

Proof. Let $I = (a_1, \dots, a_s)$. Using (†) we may assume that $J = I + (b)$. Let $C(\mathbf{a}; M)$ be the Čech-complex on M with respect to $\mathbf{a}$. Let $C(\mathbf{a}, b; M)$ be the Čech-complex on M with respect to $\mathbf{a}, b$. Note that we have a natural short exact sequence of complexes of R-modules

$$0 \to C(\mathbf{a}; M)_b[-1] \to C(\mathbf{a}, b; M) \to C(\mathbf{a}; M) \to 0.$$

Since M is a $A_n(K)$-module it is easily seen that the above map is a map of complexes of $A_n(K)$-modules. It follows that the map $H^i(C(\mathbf{a}, b; M)) \to H^i(C(\mathbf{a}; M))$ is $A_n(K)$ linear. It is easy to see that this map is $\theta^i_{J,I}(M)$. $\square$

1.6. Let $\mathfrak{a}, \mathfrak{b}$ be ideals in R and let M be an $A_n(K)$-module. Consider the Mayer-Vietoris sequence is a sequence of R-modules

$$\to H^i_{\mathfrak{a}+\mathfrak{b}}(M) \xrightarrow{\rho^i_{\mathfrak{a},\mathfrak{b}}(M)} H^i_{\mathfrak{a}}(M) \oplus H^i_{\mathfrak{b}}(M) \xrightarrow{\pi^i_{\mathfrak{a},\mathfrak{b}}(M)} H^i_{\mathfrak{a}\cap\mathfrak{b}}(M) \xrightarrow{\delta^i} H^{i+1}_{\mathfrak{a}+\mathfrak{b}}(M) \to ..$$

Then for all $i \geq 0$ the maps $\rho^i_{\mathfrak{a},\mathfrak{b}}(M)$ and $\pi^i_{\mathfrak{a},\mathfrak{b}}(M)$ are $A_n(K)$-linear.

To see this first note that since M is a $A_n(K)$-module all the above local cohomology modules are $A_n(K)$-modules. Further note that, (see [4, 15.1]),

$$\rho^i_{\mathfrak{a},\mathfrak{b}}(M)(z) = \left(\theta^i_{\mathfrak{a}+\mathfrak{b},\mathfrak{a}}(z), \theta^i_{\mathfrak{a}+\mathfrak{b},\mathfrak{b}}(z)\right),$$
$$\pi^i_{\mathfrak{a},\mathfrak{b}}(M)(x,y) = \theta^i_{\mathfrak{a},\mathfrak{a}\cap\mathfrak{b}}(x) - \theta^i_{\mathfrak{b},\mathfrak{a}\cap\mathfrak{b}}(y).$$

Using Lemma 1.5 it follows that $\rho^i_{\mathfrak{a},\mathfrak{b}}(M)$ and $\pi^i_{\mathfrak{a},\mathfrak{b}}(M)$ are $A_n(K)$-linear maps.

Remark 1.7. Infact δ^i is also $A_n(K)$-linear for all $i \geq 0$; [7]. However we will not use this fact in this paper.

1.8. Let $I_1, \ldots, I_n$ be proper ideals in R. Assume that they are pairwise co-maximal i.e., $I_i + I_j = R$ for $i \neq j$. Set $J = I_1 \cdot I_2 \cdots I_n$. Then for any R-module M we have an isomorphism of $A_n(K)$-modules

$$H^i_J(M) \cong \bigoplus_{j=1}^{n} H^i_{I_j}(M) \quad \text{for all } i \geq 0.$$

To prove this result note that I_1 and $I_2 \cdots I_n$ are co-maximal. So it suffices to prove the result for $n = 2$. In this case we use the Mayer-Vieotoris sequence of local cohomology, see 1.6, to get an isomorphism of R-modules

$$\pi^i_{I_1,I_2}(R) \colon H^i_{I_1}(R) \oplus H^i_{I_2}(R) \to H^i_{I_1 \cap I_2}(R).$$

By 1.6 we also get that $\pi^i_{I_1,I_2}(R)$ is $A_n(K)$-linear.

2. Some computations

The goal of this section is to compute the Koszul homologies $H_*(\partial_1, \ldots, \partial_n; N)$ when $N = R$ and when $N = E$ the injective hull of $R/(X_1, \ldots, X_n) = K$. It is well-known that

$$E = \bigoplus_{r_1,\ldots,r_n \geq 0} K \frac{1}{X_1 X_2 \cdots X_n X_1^{r_1} X_2^{r_2} \cdots X_n^{r_n}}.$$

Note that E has the obvious structure as a $A_n(K)$-module with

$$X_i \cdot \frac{1}{X_1 \cdots X_n X_1^{r_1} \cdots X_n^{r_n}} = \begin{cases} \frac{1}{X_1 \cdots X_n X_1^{r_1} \cdots X_i^{r_i - 1} \cdots X_n^{r_n}} & \text{if } r_i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\partial_i \cdot \frac{1}{X_1 \cdots X_n X_1^{r_1} \cdots X_n^{r_n}} = \frac{-r_i - 1}{X_1 \cdots X_n X_1^{r_1} \cdots X_i^{r_i+1} \cdots X_n^{r_n}}$$

It is convenient to introduce the following notation. For $i = 1, \cdots, n$ let $R_i = K[X_1, \ldots, X_i]$, $\mathfrak{m}_i = (X_1, \ldots, X_i)$ and let E_i be the injective hull of $R_i/\mathfrak{m}_i = K$ as a R_i-module. Set $R_0 = E_0 = K$. We prove

Lemma 2.1. *$H_0(\partial_n; E_n) \cong E_{n-1}$ and $H_1(\partial_n; E_n) = 0$ as $A_{n-1}(K)$-modules.*

Proof. Since E_n belongs to $\mathcal{B}_n(K)$ the *Bernstein class* of left $A_n(K)$ modules it follows that $H_i(\partial_n; E_n)$ (for $i = 0, 1$) belongs to $\mathcal{B}_{n-1}(K)$, the Bernstein class of left $A_{n-1}(K)$-modules [1, Chapter 1, Theorem 6.2]. We first prove $H_1(\partial_n; E_n) = 0$. Let $t \in E_n$ with $\partial_n(t) = 0$. Let

$$t = \sum_{r_1,\ldots,r_n \geq 0} t_r \frac{1}{X_1 \cdots X_n X_1^{r_1} \cdots X_n^{r_n}} \quad \text{with atmost finitely many } t_r \text{ non-zero.}$$

Notice that

$$\partial_n(t) = \sum_{r_1,\ldots,r_n\geq 0} t_r \frac{-r_n - 1}{X_1\cdots X_{n-1}X_nX_1^{r_1}\cdots X_{n-1}^{r_{n-1}}X_n^{r_n+1}}.$$

Comparing coefficients we get that if $\partial_n(t) = 0$ then $t = 0$.

For computing $H_0(\partial_n; E_n)$ we first note that as K-vector spaces

$$E_n = X \bigoplus Y;$$

where

$$X = \bigoplus_{r_1,\ldots,r_{n-1}\geq 0, r_n=0} K\frac{1}{X_1X_2\cdots X_nX_1^{r_1}X_2^{r_2}\cdots X_{n-1}^{r_{n-1}}}$$

$$Y = \bigoplus_{r_1,\ldots,r_{n-1}\geq 0, r_n\geq 1} K\frac{1}{X_1X_2\cdots X_nX_1^{r_1}X_2^{r_2}\cdots X_n^{r_n}}.$$

For $r_n \geq 1$ note that

$$\partial_n\left(\frac{1}{X_1X_2\cdots X_nX_1^{r_1}X_2^{r_2}\cdots X_n^{r_n-1}}\right) = \frac{-r_n}{X_1X_2\cdots X_nX_1^{r_1}X_2^{r_2}\cdots X_n^{r_n}}.$$

It follows that $E_n/\partial_nE_n = X$. Furthermore notice that $X \cong E_{n-1}$ as $A_{n-1}(K)$-modules. Thus we get $H_0(\partial_n; E_n) \cong E_{n-1}$. □

We now show that

Lemma 2.2. *For $c = 1, 2, \ldots, n$ we have,*

$$H_i(\partial_c, \partial_{c+1}, \cdots, \partial_n; E_n) = \begin{cases} 0 & \text{for } i > 0 \\ E_{c-1} & \text{for } i = 0 \end{cases}$$

Proof. We prove the result by induction on $t = n - c$. For $t = 0$ it is just the Lemma 2.1. Let $t \geq 1$ and assume the result for $t-1$. Let $\partial = \partial_c, \partial_{c+1}, \ldots, \partial_n$ and $\partial' = \partial_{c+1}, \ldots, \partial_n$. For each $i \geq 0$ there exist an exact sequence

$$0 \to H_0(\partial_c; H_i(\partial'; E_n)) \to H_i(\partial; E_n) \to H_1(\partial_c; H_{i-1}(\partial'; E_n)) \to 0.$$

By induction hypothesis $H_i(\partial'; E_n) = 0$ for $i \geq 1$. Thus for $i \geq 2$ we have $H_i(\partial; E_n) = 0$. Also note that by induction hypothesis $H_0(\partial'; E_n) = E_c$. So we have

$$H_1(\partial; E_n) = H_1(\partial_c; E_c) = 0 \quad \text{by Lemma 2.1.}$$

Finally again by Lemma 2.1 we have

$$H_0(\partial; E_n) = H_0(\partial_c; E_c) = E_{c-1}.$$

□

As a corollary to the above result we have

Theorem 2.3. *Let $\partial = \partial_1, \ldots, \partial_n$. Then $H_i(\partial; E_n) = 0$ for $i > 0$ and $H_0(\partial; E_n) = K$.* □

We now compute the Koszul homology $H_*(\partial; R)$. We first prove

Lemma 2.4. *$H_0(\partial_n; R_n) = 0$ and $H_1(\partial_n; R_n) = R_{n-1}$*

Proof. This is just calculus. □

The proof of the following result is similar to the proof of 2.2.

Lemma 2.5. *For $c = 1, 2, \cdots, n$ we have,*

$$H_i(\partial_c, \partial_{c+1}, \cdots, \partial_n; R_n) = \begin{cases} 0 & \text{for } i = 0, 1, \cdots, n-c \\ R_{c-1} & \text{for } i = n-c+1 \end{cases}$$

$\square$

As a corollary to the above result we have

Theorem 2.6. *Let $\partial = \partial_1, \ldots, \partial_n$. Then $H_i(\partial; R_n) = 0$ for $i < n$ and $H_n(\partial; R_n) = K$.* $\square$

3. Proof of Theorem 1

In this section we prove Theorem 1. Throughout $K \subseteq L$ where L is an algebraically closed field. We first prove:

Lemma 3.1. *Let $\mathfrak{m} = (X_1 - a_1, \cdots, X_n - a_n)$, where $a_1, \ldots, a_n \in K$, be a maximal ideal in $R = K[X_1, \ldots, X_n]$. Let $\partial = \partial_1, \ldots, \partial_n$. Then $H_i(\partial; H^n_{\mathfrak{m}}(R)) = 0$ for $i > 0$ and $H_0(\partial; H^n_{\mathfrak{m}}(R)) = K$.*

Proof. Let $U_i = X_i - a_i$ for $i = 1, \ldots, n$. Then by 1.3

$$H_i\left(\frac{\partial}{\partial U_1}, \cdots, \frac{\partial}{\partial U_n}; H^n_{\mathfrak{m}}(R)\right) \cong H_i\left(\frac{\partial}{\partial X_1}, \cdots, \frac{\partial}{\partial X_n}; H^n_{\mathfrak{m}}(R)\right)$$

for all $i \geq 0$. Thus we may assume $a_1 = a_2 = \cdots = a_n = 0$. Finally note that $H^n_{\mathfrak{m}}(R) = E$ the injective hull of $R/\mathfrak{m} = K$. So our result follows from Theorem 2.3. $\square$

We now give a proof of Theorem 1.

Proof of Theorem 1. Notice

$$A_n(L) = A_n(K) \otimes_K L$$
$$\text{and } S = L[X_1, \cdots, X_n] = R \otimes_K L.$$

So $A_n(L)$ and S are faithfully flat extensions of $A_n(K)$ and R respectively. It follows that

$$H_i\left(\partial; H^n_{IS}(S)\right) \cong H_i\left(\partial; H^n_I(R)\right) \otimes_K L \quad \text{for all } \ i \geq 0.$$

Thus we may as well assume that $K = L$ is algebraically closed. Since I is zero-dimensional we have

$$\sqrt{I} = \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_r,$$

where $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are distinct maximal ideals and $r = \sharp V(I)_L$, the number of points in $V(I)_L$. By 1.8 we have an isomorphism of $A_n(K)$-modules

$$H^j_I(R) \cong \bigoplus_{i=0}^{r} H^j_{\mathfrak{m}_i}(R) \quad \text{for all } \ j \geq 0.$$

In particular we have that

$$H_j\left(\partial; H^n_I(R)\right) = \bigoplus_{i=0}^{r} H_j\left(\partial; H^n_{\mathfrak{m}_i}(R)\right).$$

Since K is algebraically closed each maximal ideal $\mathfrak{m}$ in R is of the form $(X_1 - a_1, \ldots, X_n - a_n)$. The result follows from Lemma 3.1. $\square$

4. SOME COMPUTATIONS-II

Let $R = K[X_1, \ldots, X_n]$ and let $P = (X_1, \ldots, X_{n-1})$. The goal of this section is to compute $H_i(\partial; H_P^{n-1}(R))$ for all $i \geq 0$.

As before it is convenient to introduce the following notation. For $i = 1, \cdots, n$ let $R_i = K[X_1, \ldots, X_i]$, $\mathfrak{m}_i = (X_1, \ldots, X_i)$ and let E_i be the injective hull of $R_i/\mathfrak{m}_i = K$ as a R_i-module.

Notice that $R_{n-1} \subseteq R_n$ is a faithfully flat extension. So

$$R_n \otimes_{R_{n-1}} H^i_{\mathfrak{m}_{n-1}}(R_{n-1}) \cong H^i_{\mathfrak{m}_{n-1}R_n}(R_n) \quad \text{for all } i \geq 0.$$

Thus

$$H^{n-1}_{\mathfrak{m}_{n-1}R_n}(R_n) = E_{n-1}[X_n] = \bigoplus_{j \geq 0} E_{n-1}X_n^j.$$

We first prove the following:

Lemma 4.1. *$H_1(\partial_n; E_{n-1}[X_n]) = E_{n-1}$ and $H_0(\partial_n; E_{n-1}[X_n]) = 0$.*

Proof. Let $v \in E_{n-1}[X_n]_j$. So

$$v = \frac{c}{X_1 \cdots X_{n-1} X_1^{r_1} \cdots X_{n-1}^{r_{n-1}}} \cdot X_n^j$$

for some $c \in K$ and $r_1, \ldots, r_{n-1} \geq 0$. Notice that

$$\partial_n(v) = \begin{cases} \frac{cj}{X_1 \cdots X_{n-1} X_1^{r_1} \cdots X_{n-1}^{r_{n-1}}} \cdot X_n^{j-1} & \text{if } j \geq 1, \\ 0 & \text{if } j = 0. \end{cases}$$

It follows that $H_1(\partial_n; E_{n-1}[X_n]) = E_{n-1}$.

Let $v \in E_{n-1}[X_n]_j$ be a homogeneous element. So

$$v = \frac{c}{X_1 \cdots X_{n-1} X_1^{r_1} \cdots X_{n-1}^{r_{n-1}}} \cdot X_n^j$$

for some $c \in K$ and $r_1, \ldots, r_{n-1} \geq 0$. Let

$$u = \frac{c}{j+1} \cdot \frac{1}{X_1 \cdots X_{n-1} X_1^{r_1} \cdots X_{n-1}^{r_{n-1}}} \cdot X_n^{j+1}.$$

Notice that $\partial_n(u) = v$. Thus it follows that $H_0(\partial_n; E_{n-1}[X_n]) = 0$. $\square$

Next we prove

Lemma 4.2. *For $c = 1, 2, \ldots, n$ we have,*

$$H_i(\partial_c, \partial_{c+1}, \cdots, \partial_n; E_{n-1}[X_n]) = \begin{cases} 0 & \textit{for } i \neq 1 \\ E_{c-1} & \textit{for } i = 1. \end{cases}$$

Proof. We prove the result by induction on $t = n - c$. For $t = 0$ it is just the Lemma 4.1. Let $t \geq 1$ and assume the result for $t - 1$. Let $\partial = \partial_c, \partial_{c+1}, \ldots, \partial_n$ and $\partial' = \partial_{c+1}, \ldots, \partial_n$. For each $i \geq 0$ we have an exact sequence

$$0 \to H_0(\partial_c; H_i(\partial'; E_{n-1}[X_n])) \to H_i(\partial; E_{n-1}[X_n]) \to H_1(\partial_c; H_{i-1}(\partial'; E_{n-1}[X_n])) \to 0.$$

So $H_i(\partial; E_{n-1}[X_n]) = 0$ for $i \geq 3$ and for $i = 0$. Notice that

$$\begin{aligned} H_2(\partial; E_{n-1}[X_n]) &= H_1(\partial_c; H_1(\partial'; E_{n-1}[X_n])) \\ &= H_1(\partial_c; E_c); \text{ (by induction hypothesis).} \\ &= 0; \text{ by Lemma 2.1.} \end{aligned}$$

Similarly we have

$$\begin{aligned}H_1(\partial; E_{n-1}[X_n]) &= H_0(\partial_c; H_1(\partial'; E_{n-1}[X_n])) \\ &= H_0(\partial_c; E_c);\ \text{(by induction hypothesis).} \\ &= E_{c-1};\ \text{by Lemma 2.1.}\end{aligned}$$

$\square$

As a corollary we obtain

Theorem 4.3. *Let $R = K[X_1, \ldots, X_n]$ and let $P = (X_1, \ldots, X_{n-1})$. Let $\partial = \partial_1, \ldots, \partial_n$. Then*

$$H_i(\partial; H_P^{n-1}(R)) = \begin{cases} 0 & \text{for } i \neq 1 \\ K & \text{for } i = 1. \end{cases}$$

5. Proof of Theorem 3

In this section we prove Theorem 3. Throughout $K \subseteq L$ where L is an algebraically closed field. We first prove:

Lemma 5.1. *Let $Q = (X_1 - a_1X_n, \cdots, X_{n-1} - a_{n-1}X_n)$, where $a_1, \ldots, a_{n-1} \in K$, be a homogeneous prime ideal in $R = K[X_1, \ldots, X_n]$. Let $\partial = \partial_1, \ldots, \partial_n$. Then $H_i(\partial; H_Q^{n-1}(R)) = 0$ for $i \neq 1$ and $H_1(\partial; H_Q^{n-1}(R)) = K$.*

Proof. Let $U_i = X_i - a_iX_n$ for $i = 1, \ldots, n-1$ and let $U_n = X_n$. Then by 1.3

$$H_i\left(\frac{\partial}{\partial U_1}, \cdots, \frac{\partial}{\partial U_n}; H^n_{\mathfrak{m}}(R)\right) \cong H_i\left(\frac{\partial}{\partial X_1}, \cdots, \frac{\partial}{\partial X_n}; H^n_{\mathfrak{m}}(R)\right)$$

for all $i \geq 0$. Thus we may assume $a_1 = a_2 = \cdots = a_{n-1} = 0$. The result follows from Theorem 4.3. $\square$

We now give a proof of Theorem 3.

Proof of Theorem 3. As shown in the proof of Theorem 1 we may assume that $K = L$ is algebraically closed. We take $X_n = 0$ to be the hyperplane at infinity. After a homogeneous linear change of variables we may assume that there are no zero's of $V(I)$ in the hyperplane $X_n = 0$; see 1.3. Thus

$$\sqrt{I} = Q_1 \cap Q_2 \cap \cdots \cap Q_r$$

where $r = \sharp V(I)$ and $Q_i = (X_1 - a_{i1}X_n, \cdots, X_{n-1} - a_{i,n-1}X_n)$ for $i = 1, \ldots, r$.

We first note that $H_I^n(R) = 0$. This can be easily proved by induction on r and using the Mayer-Vieotoris sequence.

We prove the result by induction on r. For $r = 1$ the result follows from Lemma 5.1. So assume $r \geq 2$ and that the result holds for $r - 1$. Set $J = Q_1 \cap \cdots \cap Q_{r-1}$. Then $\sqrt{I} = J \cap Q_r$. Notice that $\sqrt{Q_r + J} = \mathfrak{m} = (X_1, \ldots, X_n)$. By Mayer-Vieotoris sequence and the fact that $H^n_{Q_r}(R) = H^n_J(R) = 0$ we get an exact sequence of R-modules

$$0 \to H_J^{n-1}(R) \bigoplus H_{Q_r}^{n-1}(R) \xrightarrow{\alpha} H_I^{n-1}(R) \to H^n_{\mathfrak{m}}(R) \to 0.$$

By 1.6 α is $A_n(K)$ linear. Set $C = \operatorname{coker} \alpha$. So we have an exact sequence of $A_n(K)$-modules

$$0 \to H_J^{n-1}(R) \bigoplus H_{Q_r}^{n-1}(R) \xrightarrow{\alpha} H_I^{n-1}(R) \to C \to 0.$$

Claim: $C \cong H^n_{\mathfrak{m}}(R)$ as $A_n(K)$-modules.

First suppose the claim is true. Then note that the result follows from induction hypothesis and Lemma's 3.1, 5.1.

It remains to prove the claim. Note that $C \cong H^n_{\mathfrak{m}}(R)$ as R-modules. In particular

$$\operatorname{soc}_R(C) = \operatorname{Hom}_R(R/\mathfrak{m}, C) \cong \operatorname{Hom}_R(R/\mathfrak{m}, H^n_{\mathfrak{m}}(R)) \cong K.$$

Let e be a non-zero element of $\operatorname{soc}_R(C)$. Consider the map

$$\begin{aligned} \phi \colon A_n(K) &\to C \\ d &\mapsto de. \end{aligned}$$

Clearly ϕ is $A_n(K)$-linear. Since $\phi(A_n(K)\mathfrak{m}) = 0$ we get an $A_n(K)$-linear map

$$\overline{\phi} \colon \frac{A_n(K)}{A_n(K)\mathfrak{m}} \to C.$$

Note that $A_n(K)/A_n(K)\mathfrak{m} \cong H^n_{\mathfrak{m}}(R)$ as $A_n(K)$-modules.

To prove that $\overline{\phi}$ is an isomorphism, note that $\overline{\phi}$ is R-linear. Since $\overline{\phi}$ induces an isomprhism on socles we get that $\overline{\phi}$ is injective. As $H^n_{\mathfrak{m}}(R)$ is an injective R-module and $\overline{\phi}$ is injective R-linear map we have that $C \cong \operatorname{image} \overline{\phi} \oplus \operatorname{coker} \overline{\phi}$ as R-modules. Set $N = \operatorname{coker} \overline{\phi}$. Note that $\operatorname{soc}_R(N) = 0$. Also note that as R-module C is supported only at $\mathfrak{m}$. So N is supported only at $\mathfrak{m}$. Since $\operatorname{soc}_R(N) = 0$ we get that $N = 0$. So $\overline{\phi}$ is surjective. Thus $\overline{\phi}$ is an $A_n(K)$-linear isomorphism of $A_n(K)$-modules. ☐

6. Proof of Theorem 5

In this section we prove Theorem 5. We first prove

Lemma 6.1. *Let f be a non-constant squarefree polynomial in $R = K[X_1, \ldots, X_n]$. Let $\partial = \partial_1, \ldots, \partial_n$. Then $H_n(\partial; R_f) = K$.*

Proof. Note that

$$H_n(\partial; R_f) = \{v \in R_f \mid \partial_i v = 0 \text{ for all } i = 1, \ldots, n\}.$$

Clearly if $v \in R_f$ is a constant then $\partial_i v = 0$ for all $i = 1, \ldots, n$. By a linear change in variables we may assume that $f = X_n^s + \text{lower terms in } X_n$. Note that by 1.3 the Koszul homology does not change.

Suppose if possible there exists a non-constant $v = a/f^r \in H_n(\partial; R_f)$ where f does not divide a if $r \geq 1$. Note that if $r = 0$ then $v \in H_n(\partial; R) = K$. So v is a constant. So assume $r \geq 1$. Since $\partial_n(v) = 0$ we get $f\partial_n(a) = ra\partial_n(f)$.

Since f is squarefree we have $f = f_1 \cdots f_m$ where f_i are distinct irreducible polynomials. As f is monic in X_n we have that f_i is monic in X_n for each i.

Since $f\partial_n(a) = ra\partial_n(f)$ we have that f_i divides $a\partial_n(f)$ for each i. Note that if f_i divides $\partial_n(f)$ then f_i divides $f_1 \cdots f_{i-1}\partial_n(f_i) \cdot f_{i+1} \cdots f_m$. Therefore f_i divides $\partial_n(f_i)$ which is easily seen to be a contradiction since f_i is monic in X_n. Thus f_i divides a for each $i = 1, \ldots, m$. Therefore f divides a, which is a contradiction. Thus $H_n(\partial; R_f)$ only consists of constants. ☐

We now give a proof of Theorem 5.

Proof of Theorem 5. We prove the result by induction on number of generators of I. We first consider the case when $I = (f)$ is a principal ideal. Since we are taking

local cohomology we may assume that I is radical ideal; so f is squarefree. We have an exact sequence

$$0 \to R \to R_f \to H^1_I(R) \to 0.$$

Notice $H_n(\partial, R) = H_n(\partial; R_f) = K$ and $H_{n-1}(\partial, R) = 0$ (see Theorem 2.6 and Lemma 6.1). So we get $H_n(\partial, H^1_I(R)) = 0$.

Let $s \geq 2$ and assume the result holds if I is generated by $s-1$ elements. Let $I = (f_1, \ldots, f_s)$. Let $J = (f_1, \ldots, f_{s-1})$. By Mayer-Vieotoris we have an exact sequence of R-modules

$$0 \to H^1_I(R) \to H^1_J(R) \bigoplus H^1_{(f_s)}(R) \xrightarrow{\alpha} H^1_{Jf_s}(R).$$

By 1.6 the above sequence is a sequence of $A_n(K)$-modules. Let $C = \text{image } \alpha$. So we have an exact sequence of $A_n(K)$-modules

$$0 \to H^1_I(R) \to H^1_J(R) \bigoplus H^1_{(f_s)}(R) \to C \to 0.$$

The long exact sequence of Koszul homology and the induction hypothesis yields the result. □

7. proof of Theorem 6

In this section we prove Theorem 6.

7.1. Let A be a Noetherian ring, I an ideal in A and let M be an A-module, not necessarily finitely generated. Set

$$\Gamma_I(M) = \{m \in M \mid I^s m = 0 \text{ for some } s \geq 0\}.$$

The following result is well-known. For lack of a suitable reference we give sketch of a proof here. When M is finitely generated, for a proof of the following result see [3, Proposition 3.13].

Lemma 7.2. *[with hyotheses as above]*

$$\operatorname{Ass}_A \frac{M}{\Gamma_I(M)} = \{P \in \operatorname{Ass}_A M \mid P \not\supseteq I\}$$

Proof. (sketch) Note that if $P \in \operatorname{Ass}_A \Gamma_I(M)$ then $P \supseteq I$. It follows that if $P \in \operatorname{Ass}_A M$ and $P \not\supseteq I$ then $P \in \operatorname{Ass}_A M/\Gamma_I(M)$.

It can be easily verified that if $P \in \operatorname{Ass}_A M/\Gamma_I(M)$ then $P \not\supseteq I$. Also note that if $P \not\supseteq I$ then $\Gamma_I(M)_P = 0$. Thus

$$M_P \cong \left(\frac{M}{\Gamma_I(M)}\right)_P \quad \text{if } P \not\supseteq I.$$

The result follows. □

We now give

Proof of Theorem 6. First consider the case when K is algebraically closed. Set

$$\operatorname{Ass}_A(M) = \operatorname{mIso}_R(M) \sqcup \left(\bigcup_{i=1}^{s} V(P_i) \cap \operatorname{Ass}_A(M)\right).$$

Here $P_1, \ldots, P_s$ are minimal primes of M which are not maximal ideals.

Set $I = P_1 P_2 \cdots P_s$. Note that $\Gamma_I(M)$ is a $A_n(K)$-submodule of M. Set $N = M/\Gamma_I(M)$. By Lemma 7.2 we get that

$$\begin{aligned}\operatorname{Ass}_R N &= \{P \in \operatorname{Ass}_R M \mid P \not\supseteq I\} \\ &= \operatorname{mIso}(M).\end{aligned}$$

Let $\operatorname{mIso}(M) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$. Set $J = \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_r$. Since $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are comaximal we get by 1.8 that as $A_n(K)$-modules

$$\Gamma_J(N) = \Gamma_{\mathfrak{m}_1}(N) \oplus \cdots \oplus \Gamma_{\mathfrak{m}_r}(N).$$

Set $E = N/\Gamma_J(N)$. By Lemma 7.2 we get that $\operatorname{Ass}_R E = \emptyset$. So $E = 0$. Thus

$$N = \Gamma_{\mathfrak{m}_1}(N) \oplus \cdots \oplus \Gamma_{\mathfrak{m}_r}(N).$$

Note that

$$\Gamma_{\mathfrak{m}_i}(N) = E_R(R/\mathfrak{m}_i)^{s_i} = H^n_{\mathfrak{m}_i}(R)^{s_i} \quad \text{for some } s_i \geq 1.$$

Since K is algebraiclly closed we have that for each $i = 1, \ldots, r$ the maximal ideal $\mathfrak{m}_i = (X_1 - a_{i1}, \ldots, X_n - a_{in})$ for some $a_{ij} \in K$. It follows from Lemma 3.1 that

$$H_i(\partial; N) = 0 \text{ for } i \geq 1$$

$$\dim_K H_0(\partial; N) = \sum_{i=1}^{r} s_i.$$

The exact sequence $0 \to \Gamma_I(M) \to M \to N \to 0$ yields an exact sequence of Koszul homologies

$$0 \to H_0(\partial; \Gamma_I(M)) \to H_0(\partial; M) \to H_0(\partial; N) \to 0;$$

since $H_1(\partial; N) = 0$. The result follows. So we have proved the result when K is algebraically closed.

Now consider the case when K is *not* algebraically closed. Let $L = \overline{K}$ the algebraic closure of K. Note that $S = L[X_1, \ldots, X_n] = R \otimes_K L$ and $A_n(L) = A_n(K) \otimes_K L$. Further notice that $M \otimes_K L$ is a holonomic $A_n(L)$-module. Also note that $M \otimes_R S = M \otimes_K L$.

Claim-1 : $\sharp \operatorname{mIso}_S(M \otimes_R S) \geq \sharp \operatorname{mIso}_R(M)$.

We assume the claim for the moment. Note that $H_0(\partial, M) \otimes_K L = H_0(\partial, M \otimes_K L)$. So

$$\dim_K H_0(\partial, M) = \dim_L H_0(\partial, M \otimes_K L) \geq \sharp \operatorname{mIso}_S(M \otimes_R S) \geq \sharp \operatorname{mIso}_R(M).$$

The result follows.

It remains to prove Claim-1. By Theorem 23.2(ii) of [6] we have

$$(\dagger) \qquad \operatorname{Ass}_S(M \otimes_R S) = \bigcup_{P \in \operatorname{Ass}_R(M)} \operatorname{Ass}_S\left(\frac{S}{PS}\right).$$

Suppose $\mathfrak{m}$ is an isolated maximal prime of M. Notice $S/\mathfrak{m}S$ has finite length. It follows that

$$\sqrt{\mathfrak{m}S} = \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_r;$$

for some maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_r$ of S.

Claim-2 : $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_r \in \operatorname{mIso}_S(M \otimes_R S)$.

Note that Claim-2 implies Claim-1. It remains to prove Claim-2.

Suppose if possible some $\mathfrak{m}_i \notin \operatorname{mIso}_S(M \otimes_R S)$. Then there exist $Q \subsetneqq \mathfrak{m}_i$ and $Q \in \operatorname{Ass}_S(M \otimes_R S)$. Note that Q is not a maximal ideal in S. By (†) we have that

$$Q \in \operatorname{Ass}_S\left(\frac{S}{PS}\right) \quad \text{for some } P \in \operatorname{Ass}_R(M).$$

Notice that as Q is not a maximal ideal in S we have that P is not a maximal ideal in R. Also note that by Theorem 23.2(i) of [6] we have

$$P = Q \cap R \subseteq \mathfrak{m}_i \cap R = \mathfrak{m}.$$

Thus $\mathfrak{m}$ is not an isolated maximal prime of M, a contradiction. □

An application of Theorem 5 is the following result:

Corollary 7.3. *Let I be an ideal of height $\leq n-2$ in R. Then*

$$\sharp \operatorname{Ass}_R H^{n-1}_I(R) \leq \dim_K H_0\left(\partial, H^{n-1}_I(R)\right).$$

Proof. We first show that $M = H^{n-1}_I(R)$ is supported only at maximal ideals of R. As M is I-torsion it follows that any $P \in \operatorname{Supp}(M)$ contains I.

We first show that if $\operatorname{ht} P \leq n-2$ then $P \notin \operatorname{Supp}(M)$. Note $M_P = H^{n-1}_{IR_P}(R_P) = 0$ by Grothendieck vanishing theorem as $\dim R_P = \operatorname{ht} P \leq n-2$. So $P \notin \operatorname{Supp}(M)$.

Next we prove that $\operatorname{ht} P = n-1$ then $P \notin \operatorname{Supp}(M)$. Let $\widehat{R_P}$ be the completion of R_P with respect to its maximal ideal. Note $M_P \otimes_{R_P} \widehat{R_P} = H^{n-1}_{I\widehat{R_P}}(\widehat{R_P}) = 0$ by Hartshorne-Lichtenbaum Vanishing theorem as $I\widehat{R_P}$ is not $P\widehat{R_P}$-primary. As $\widehat{R_P}$ is a faithfully flat R_P algebra we have $M_P = 0$.

Thus M is supported at only maximal ideals of R. It follows that $\operatorname{Ass}_A(M) = \operatorname{mIso}_R(M)$. The result now follows from Theorem 5. □

8. proof of Theorem 2

In this section we prove Theorem 2. Set $R_{n-1} = K[X_1, \ldots, X_{n-1}]$.

We begin by the following result on vanishing (and non-vanishing) of Koszul homology of a simple $A_n(K)$-module. If M is a simple $A_n(K)$-module then it is well-known that $\operatorname{Ass}_R(M)$ consists of a singleton set.

Theorem 8.1. *Let M be a simple $A_n(K)$-module and assume $\operatorname{Ass}_R(M) = \{P\}$. Set $Q = P \cap R_{n-1}$. Then*

$$H_0(\partial_n; M) = 0 \implies P = QR,$$
$$H_1(\partial_n; M) \neq 0 \implies P = QR.$$

To prove the above theorem we need a criterion for an ideal I to be equal to $(I \cap R_{n-1})R$. This is provided by the following:

Lemma 8.2. *Let I be an ideal in R. Set $J = I \cap R_{n-1}$. Then the following are equivalent:*

(1) $\partial_n(I) \subseteq I$.
(2) $I = JR$.
(3) *Let $\xi \in I$. Let $\xi = \sum_{j=0}^{m} c_j X_n^j$ where $c_j \in R_{n-1}$ for $j = 0, \ldots, m$. Then $c_j \in I$ for each j.*

Proof. We first prove (1) $\implies$ (3). Let $\xi \in I$. Let $\xi = \sum_{j=0}^{m} c_j X_n^j$ where $c_j \in R_{n-1}$ for $j = 0, \dots, m$. Notice $\partial_n^m(\xi) = m! c_m$. So $c_m \in I$. Thus $\xi - c_m X_n^m \in I$. Iterating we obtain that $c_j \in I$ for all j.

Notice that (3) $\implies$ (1) is trivial. We now show (3) $\implies$ (2). Let $\xi \in I$. Let $\xi = \sum_{j=0}^{m} c_j X_n^j$ where $c_j \in R_{n-1}$ for $j = 0, \dots, m$. By hypothesis $c_j \in I$ for each j. Notice $c_j \in I \cap R_{n-1} = J$. Thus $I \subseteq JR$. The assertion $JR \subseteq I$ is trivial. So $I = JR$.

Finally we prove that (2) $\implies$ (3). If $b \in J$ and $r \in R$ then notice that if $br = \sum_{j=0}^{m} c_j X_n^j$ where $c_j \in R_{n-1}$ for $j = 0, \dots, m$ then each $c_j \in J$. As $I = JR$ each $\xi \in I$ is a finite sum $b_1 r_1 + \cdots + b_s r_s$ where $b_i \in J$ and $r_i \in R$. The assertion follows. $\square$

The following corollary is useful.

Corollary 8.3. *Let P be a prime ideal in R and let I be an ideal in R with $\sqrt{I} = P$. If $\partial_n(I) \subseteq I$ then $P = (P \cap R_{n-1})R$.*

Proof. Set $Q = P \cap R_{n-1}$. Let $\xi \in P$. Let $\xi = \sum_{j=0}^{m} c_j X_n^j$ where $c_j \in R_{n-1}$ for $j = 0, \dots, m$. Notice $\xi^s \in I$ for some $s \geq 1$. Also $\xi^s = c_m^s X_n^{sm} + ..$ lower terms in X_n. By Lemma 8.2 we get that $c_m^s \in I$. It follows that $c_m \in P$. Thus $\xi - c_m X_n^m \in P$. Iterating we obtain that $c_j \in P$ for all j. So by Lemma 8.2 we get that $P = QR$. $\square$

We now give

Proof of Theorem 8.1. First suppose $H_0(\partial_n, M) = 0$. Let $a \in M$ with $P = (0\colon a)$. Say $\partial_n b = a$. Set $I = (0\colon b)$.

We first claim that $I \subseteq P$. Let $\xi \in I^2$. Notice $\partial_n \xi = \xi \partial_n + \partial_n(\xi)$. Also note that $\partial_n(\xi) \in I$. Thus we have that $\partial_n \xi b = \xi a + \partial_n(\xi) b$. Thus $\xi a = 0$. So $\xi \in P$. Thus $I^2 \subseteq P$. As P is a prime ideal we get that $I \subseteq P$.

Next we claim that $\partial_n(I) \subseteq I$. Let $\xi \in I$. We have $\partial_n \xi b = \xi a + \partial_n(\xi) b$. So $\partial_n(\xi) b = 0$. Thus $\partial_n(\xi) \in I$.

Since M is simple we have that $M = A_n(K)a$. So $b = da$ for some $d \in A_n(K)$. It can be easily verified that there exists $s \geq 1$ with $P^s d \subseteq A_n(K)P$. It follows that $P^s \subseteq I$. Thus $\sqrt{I} = P$. The result follows from 8.3.

Next suppose $H_1(\partial_n; P) \neq 0$. Say $a \in \ker \partial_n$ is non-zero. Set $J = (0\colon a)$. Let $\xi \in J$. Notice $\partial_n \xi a = \xi \partial_n a + \partial_n(\xi) a$. Thus $\partial_n(\xi) a = 0$. Thus $\partial_n(J) \subseteq J$.

By hypothesis M is simple and $\operatorname{Ass}_R(M) = \{P\}$. Now $\Gamma_P(M)$ is a non-zero $A_n(K)$-submodule of M. As M is simple we have that $M = \Gamma_P(M)$. Thus $P^s a = 0$ for some $s \geq 1$. Thus $P^s \subseteq J$. Also note that for any R-module E the maximal elements in the set $\{(0\colon e) \mid e \neq 0\}$ are associate primes of E. Thus $J = (0\colon a) \subseteq P$. Therefore $\sqrt{J} = P$. The result follows from 8.3. $\square$

We now extend Lemma 8.2 and Corollary 8.3.

Lemma 8.4. *Let I be an ideal in R. Set $R_c = K[X_1, \dots, X_c]$ and $J = I \cap R_c$. Then the following are equivalent:*

(1) *$\partial_j(I) \subseteq I$, for $j = c+1, \dots, n$.*
(2) *$I = JR$.*

Proof. We first prove that (2) $\implies$ (1). Let $\xi \in I$. Since $I = JR$ we have that

$$\xi = \sum_v r_v X_{c+1}^{v_{c+1}} \cdots X_n^{v_n} \quad \text{where } r_v \in J \text{ for all } \ v.$$

It follows that $\partial_j(I) \subseteq I$, for $j = c+1, \ldots, n$.

For the converse set $R_i = K[X_1, \ldots, X_i]$ and $J_i = I \cap R_i$ for $i = c, \ldots, n-1$. Since $\partial_n(I) \subseteq I$, by Lemma 8.2 we have that $I = J_{n-1}R$. Since $\partial_{n-1}(I) \subseteq I$ we have that $\partial_{n-1}(J_{n-1}) \subseteq J_{n-1}$. So again by 8.2 we have that $J_{n-1} = J_{n-2}R_{n-1}$. In particular we have that $I = J_{n-2}R$. Iterating this procedure we get the required result i.e., $I = JR$. ☐

Corollary 8.5. *Let P be a prime ideal in R and let I be an ideal in R with $\sqrt{I} = P$. Set $R_c = K[X_1, \ldots, X_c]$ and $Q = P \cap R_c$. If $\partial_j(I) \subseteq I$, for $j = c+1, \ldots, n$. then $P = QR$.*

Proof. By 8.3 we get that $P = (P \cap R_i')R$ for $i = c+1, \ldots, n$ where $R_i' =$ polynomial ring over K in the variables $\{X_1, \ldots, X_n\} \setminus \{X_i\}$. By 8.2 we have that $\partial_i(P) \subseteq P$ for $i = c+1, \ldots, n$. The result follows from 8.4. ☐

Corollary 8.6. *Let I be a non-zero ideal in R. Also assume $I \neq R$. Then there exists i such that $\partial_i(I) \nsubseteq I$.*

Proof. Suppose if possible $\partial_i(I) \subseteq I$ for all $i = 1, \ldots, n$. Let $J = I \cap K$. Then by Lemma 8.4 we get that $I = JR$. It follows that $I = 0$ or R, a contradiction. ☐

Remark 8.7. Let P be a prime ideal in R. Set $Q = P \cap R_{n-1}$. Then it can be easily seen that

$$\operatorname{ht}_R P - 1 \leq \operatorname{ht}_{R_{n-1}} Q \leq \operatorname{ht}_R P.$$

Furthermore $\operatorname{ht}_{R_{n-1}} Q = \operatorname{ht}_R P$ if and only if $P = QR$.

Remark 8.8. Let M be a holonomic $A_n(K)$-module. Assume M is I-torsion. Set $J = I \cap R_{n-1}$. Then for $i = 0, 1$ the Koszul homology modules $H_i(\partial_n, M)$ are J-torsion holonomic $A_{n-1}(K)$-modules. For holonomicity see 8.8. Also note the sequence

$$0 \to H_1(\partial_n, M) \to M \xrightarrow{\partial_n} M \to H_0(\partial_n, M) \to 0$$

is an exact sequence of $A_{n-1}(K)$-modules. It follows that $H_i(\partial_n, M)$ are J-torsion for $i = 0, 1$.

The following result is an essential ingredient in the proof of Theorem 2. Let us recall that a holonomic $A_n(K)$-module M has a composition series, see [1, 1.5.3] . We denote lenght of M as an $A_n(K)$-module by $\ell_{A_n(K)}(M)$.

Proposition 8.9. *Let P be a height $n-1$ prime in R. Let M be a non-zero holonomic $A_n(K)$-module. Assume M is P-torsion. Set $Q = P \cap R_{n-1}$. Assume $P \neq QR$. Then*

(1) $H_1(\partial_n; M) = 0$.
(2) $\ell_{A_{n-1}(K)}(H_0(\partial_n; M)) \geq \ell_{A_n(K)}(M)$.
(3) $H_0(\partial_n; M)$ *is Q-torsion.*

Proof. Let $c = \ell_{A_n(K)}(M)$. So we have a composition series

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_c = M.$$

For $i = 1, \ldots, c$, $N_i = V_i/V_{i-1}$ are simple holonomic $A_n(K)$-modules.

Fix i. Let $\operatorname{Ass}_R(N_i) = \{W_i\}$. As N_i is a subquotient of M we get that N_i is P-torsion. So $W_i \supseteq P$. Thus either $W_i = P$ or W_i is a maximal ideal of R. If $W_i = P$ then by hypothesis $W_i \neq (W_i \cap R_{n-1})R$. If W_i is a maximal ideal of R then by Remark 8.7 it follows that $W_i \neq (W_i \cap R_{n-1})R$. Thus by Theorem 8.1 we get that

$$(*) \qquad H_1(\partial_n; N_i) = 0 \quad \text{and} \quad H_0(\partial_n; N_i) \neq 0.$$

(1) and (2): We first note that we have an exact sequence

$$0 \to V_1 \to V_2 \to N_2 \to 0.$$

Notice $V_1 = N_1$. By (*) we get

$$H_1(\partial_n; V_2) = 0 \quad \text{and} \quad \ell_{A_{n-1}(K)}\left(H_0(\partial_n; V_2)\right) \geq 2.$$

An easy induction yields

$$H_1(\partial_n; M) = 0 \quad \text{and} \quad \ell_{A_{n-1}(K)}\left(H_0(\partial_n; M)\right) \geq c.$$

(3) By 8.8, $H_0(\partial_n, M)$ is Q-torsion. $\square$

Theorem 2 is an easy corollary of the following more general result:

Theorem 8.10. *Let P be a height $n-1$ prime in R with $n \geq 2$. Let M be a non-zero P-torsion holonomic $A_n(K)$-module. Then*

$$H_j(\partial; M) = 0, \quad for\ j \geq 2.$$

Proof. Set $M_n = M$ and $P_n = P$. Also set $R_i = K[X_1, \ldots, X_i]$. By Proposition 8.6, $\partial_i(P_n) \nsubseteq P_n$ for some i. After relabeling we may assume $i = n$. Set $P_{n-1} = P \cap R_{n-1}$. As $\partial_n(P_n) \nsubseteq P_n$, by 8.2 we get $P_n \neq P_{n-1}R$. By Proposition 8.9 we have that

$$H_1(\partial_n; M_n) = 0 \quad \text{and} \quad H_0(\partial_n; M_n) \neq 0.$$

Set $M_{n-1} = H_0(\partial_n, M_n)$ a holonomic $A_{n-1}(K)$-module. Also M_{n-1} is P_{n-1}-torsion. By remark 8.7 we get that $\operatorname{ht} P_{n-1} = n-2$ since $P_n \neq P_{n-1}R_n$.

If $n - 2 \geq 1$ then we repeat the process of the previous paragraph. So after a possible relabeling we have

$$H_1(\partial_{n-1}; M_{n-1}) = 0 \quad \text{and} \quad H_0(\partial_{n-1}; M_{n-1}) \neq 0.$$

Therefore by Lemma 1.2 we get that

$$H_i(\partial_{n-1}, \partial_n; M_n) = 0 \text{ for } i \geq 1,$$
$$H_0(\partial_{n-1}, \partial_n; M_n) = H_0(\partial_{n-1}, M_{n-1}) \neq 0.$$

Set $M_{n-2} = H_0(\partial_{n-1}; M_{n-1}) = H_0(\partial_{n-1}, \partial_n; M_n)$ and $P_{n-2} = P_{n-1} \cap R_{n-2}$.

We iterate this process to obtain (after relabeling) a non-zero holonomic $A_1(K)$ module $M_1 = H_0(\partial_2, \cdots, \partial_n; M_n)$. We also get $H_i(\partial_2, \cdots, \partial_n; M_n) = 0$ for $i \geq 1$.

Note that P_1 has height 0, so $P_1 = 0$. Thus our process ends. Trivially we have that $H_i(\partial_1, M_1) = 0$ for $i \geq 2$. So by Lemma 1.2 we have $H_i(\partial, M) = 0$ for $i \geq 2$. $\square$

We now give

Proof of Theorem 2. Note that $M = H^{n-1}_P(R)$ is a holonomic $A_n(K)$-module. Also note that $M_P = H^{n-1}_{PR_P}(R_P)$ is non-zero by Grothendieck non-vanishing theorem. So $M \neq 0$. Clearly M is P-torsion. Thus by Theorem 8.10 we get that $H_i(\partial, M) = 0$ for $i \geq 2$. $\square$

9. proof of Theorem 4

In this section we prove Theorem 4. We need some preliminaries on graded $A_n(K)$-modules.

9.1. We first note that $A_n(K)$ has a natural structure of a $\mathbb{Z}$-graded ring. For each $i = 1, \ldots, n$ we give the variables X_i degree 1 and ∂_i degree -1. The ring R with the usual grading is clearly a graded $A_n(K)$-module. If M is a $\mathbb{Z}$-graded $A_n(K)$-module and $f \in R$ is homogeneous then M_f is clearly a graded $A_n(K)$-module. Further note that if f, g are homogeneous elements in R then the natural map $M_f \to M_{fg}$ is a homogeneous map(of degree 0) of graded $A_n(K)$-modules.

9.2. Let I be a homogeneous ideal in R. We choose a homogeneous generating set $(f_1, \ldots, f_r)$ of I. By computing the local cohomology modules $H^i_I(R)$ via the Čech-complex it follows that $H^i_I(R)$ are graded $A_n(K)$-modules for all $i \geq 0$.

9.3. The inclusion $R \subseteq A_n(K)$ is an inclusion of graded rings. Thus every graded $A_n(K)$-module M is a graded R-module. Let $S = K[\partial_1, \ldots, \partial_n]$. For each $i = 1, \ldots, n$ we give ∂_i degree -1. Then S is also a graded subring of $A_n(K)$. It follows that if M is a graded $A_n(K)$-module then each Koszul homology module $H_i(\partial, M)$ is a graded K-vector space. More generally $H_i(\partial_{c+1}, \ldots, \partial_n, M)$ is a graded $A_c(K)$-module for each $i \geq 0$.

Definition 9.4. A non-zero graded $A_n(K)$-module M is said to be *-simple if it has no proper graded submodules.

Example 9.5. Let $\mathfrak{m} = (X_1, \ldots, X_n)$. Let E be the injective hull of $k = R/\mathfrak{m}$ as a R-module. Then it is well-known that E is a graded $A_n(K)$-module. It is also well-known that E is a simple $A_n(K)$-module. So E is a *-simple $A_n(K)$-module.

Remark 9.6. I do not know o an example of a *-simple $A_n(K)$-module which is not simple(in the absolute sense).

Proposition 9.7. *Let M be a *-simple $A_n(K)$-module. Then* $\operatorname{Ass}_R(M) = \{P\}$ *for some homogeneous prime ideal P in R. Furthermore $M = \Gamma_P(M)$.*

Proof. As M is a graded R-module all its associated primes are homogeneous, see [2, 1.5.6]. Let P be a maximal element in $\operatorname{Ass}_R M$. Set $N = \Gamma_P(M)$. Then note that $N \neq 0$, $A_n(K)$ submodule of M and $\operatorname{Ass}_R(N) = \{P\}$. It is also clear that N is a graded $A_n(K)$-module. As M is *-simple we have $N = M$. □

We now give a graded version of Theorem 8.1.

Theorem 9.8. *Let M be a *-simple $A_n(K)$-module and assume* $\operatorname{Ass}_R(M) = \{P\}$. *Set $Q = P \cap R_{n-1}$. Then*

$$H_0(\partial_n; M) = 0 \implies P = QR,$$
$$H_1(\partial_n; M) \neq 0 \implies P = QR.$$

The proof is analogus to the proof of Theorem 8.1. So we will only give a sketch of a proof.

Proof. First suppose $H_0(\partial_n; M) = 0$. By 9.7, P is a homogeneous prime. By [2, 1.5.6] $P = (0\colon a)$ for some homogeneous element a in M. The rest of the proof is nearly identical to that of Theorem 8.1. The only thing to note that $M = A_n(K)a$ as M is *-simple and a is homogeneous.

Next suppose $H_1(\partial_n; M) = 0$. The proof in this case early identical to that of Theorem 8.1. The only thing to note that $M = \Gamma_P(M)$, since $\Gamma_P(M)$ is a non-zero graded submodule of M and M is *-simple. $\square$

The following Proposition is a "*"-version of existence of a composition series of a module.

Proposition 9.9. *Let M be a non-zero holonomic $A_n(K)$-module. Assume that M is graded as a $A_n(K)$-module. Then there exists a filtration of graded submodules*

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_c = M,$$

such that for $i = 1, \ldots, c$ the module V_i/V_{i-1} is *-simple.

Proof. (Sketch) Note that M is Artinan and Noetherian as a $A_n(K)$-module. Let

$$\mathcal{C} = \{N \mid N \text{ is a non-zero graded submodule of } M\}.$$

The set $\mathcal{C}$ is non-empty and as M is Artin it has a minimal element, say V_1. Clearly V_1 is *-simple.

If $V_1 = M$ then stop. Otherwise consider M/V_1 and repeat the process.

The process ends since M is Noetherian as a $A_n(K)$-module. $\square$

9.10. For $i = 1, \cdots, n$ let $R_i = K[X_1, \ldots, X_i]$, $\mathfrak{m}_i = (X_1, \ldots, X_i)$ and let E_i be the injective hull of $R_i/\mathfrak{m}_i = K$ as a R_i-module. Set $R_0 = E_0 = K$.

The following Proposition is an essential ingredient in the proof of Theorem 4.

Proposition 9.11. *Let P be a height $n-2$ prime in R. Let M be a graded holonomic $A_n(K)$-module. Assume M is P-torsion and $P \neq QR$. Then*

(1) *$H_i(\partial_n; M)$ is a graded holonomic $A_{n-1}(K)$-module for $i = 0, 1$.*
(2) *$H_0(\partial_n; M)$ is Q-torsion.*
(3) *$H_1(\partial_n; M)$ is $\mathfrak{m}_{n-1}$-torsion.*
(4) *$H_1(\partial_n; M) = 0$ or $H_0(\partial_n; M) \cong \bigoplus_{i=1}^{c} E_{n-1}(-a_i)$ as graded $A_{n-1}(K)$-modules.*

Proof. (1) This follows from 1.1 and 9.3.

(2) This follows from 8.8.

(3)Let $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_c = M$ be a filtration of M by graded submodules with $N_i = V_i/V_{i-1}$ *-simple for $i = 1, \ldots, c$.

Fix i. Let $\operatorname{Ass}_R N_i = \{W_i\}$. By 9.7, W_i is a homogeneous prime ideal in R. As N_i is a subquotient of M, we get that N_i is P-torsion. So $W_i \supseteq P$. Thus $W_i = P$ or W_i is a height $n-1$ homogeneous prime in R or $W_i = \mathfrak{m}_n$.

Fix i. If $W_i = P$ then by hypothesis $W_i \neq (W_i \cap R_{n-1})R_n$. If $W_i = \mathfrak{m}_n$ then clearly $W_i \neq (W_i \cap R_{n-1})R_n$. If W_i is a height $n-1$ homogeneous prime ideal in R with $W_i = (W_i \cap R_{n-1})R_n$, then note that $W_i \cap R_{n-1}$ is a graded prime ideal of R_{n-1} with height $n-1$, see Remark 8.7. So $W_i \cap R_{n-1} = \mathfrak{m}_{n-1}$. So $W_i = \mathfrak{m}_{n-1}R$. Conversely if $W_i \neq (W_i \cap R_{n-1})R_n$ then $W_i = (W_i \cap R_{n-1})R_n$.

Fix i. Note that $H_1(\partial_n; N_i) = 0$ if $W_i \neq \mathfrak{m}_{n-1}R$, see 9.8. If $W_i = \mathfrak{m}_{n-1}R$ then as N_i is $\mathfrak{m}_{n-1}R$-torsion we get that $H_1(\partial_n; N_i)$ is $\mathfrak{m}_{n-1}$-torsion. Thus in any case $H_1(\partial_n; N_i)$ is $\mathfrak{m}_{n-1}$-torsion.

We prove that $H_1(\partial_n; V_i)$ is $\mathfrak{m}_{n-1}$-torsion by induction on i. For $i = 1, V_1 = N_1$. So $H_1(\partial_n; V_1)$ is $\mathfrak{m}_{n-1}$-torsion. If we assume that $H_1(\partial_n; V_{i-1})$ is $\mathfrak{m}_{n-1}$-torsion then using the exact sequence

$$0 \to V_{i-1} \to V_i \to N_i \to 0,$$

we get an exact sequence
$$0 \to H_1(\partial_n; V_{i-1}) \to H_1(\partial_n; V_i) H_1(\partial_n; N_i).$$
It follows that $H_1(\partial_n, V_i)$ is $\mathfrak{m}_{n-1}$-torsion.

(4) This is well-known. $\square$

Theorem 4 is a corollary of the following more general result:

Theorem 9.12. *Let P be a height $n-2$ homogeneous prime in R with $n \geq 3$. Let M be a non-zero P-torsion graded holonomic $A_n(K)$-module. Then*
$$H_i(\partial; M) = 0, \quad \textit{for } i \geq 3.$$

Proof. Set $M_n = M$ and $P_n = P$. By Corollary 8.6, $\partial_i(P_n) \nsubseteq P_n$ for some i. After relabeling we may assume $i = n$. Set $P_{n-1} = P_n \cap R_{n-1}$. Notice $P_n \neq P_{n-1}R$. So by Proposition 9.11, $M_{n-1} = H_0(\partial_n; M_n)$ is a graded holonomic P_{n-1}-torsion A_{n-1}-module and $L_{n-1} = H_1(\partial_n; M_n$ is zero or some copies of E_{n-1}.

By remark 8.7, $\operatorname{ht} P_{n-1} = n-3$. If $n = 3$ then stop. Otherwise continue the process. After relabeling we may assume $\partial_{n-1}(P_{n-1}) \nsubseteq P_{n-1}$. Notice $P_n \neq P_{n-1}R$. So by Proposition 9.11, $M'_{n-2} = H_0(\partial_n; M_{n-1})$ is graded holonomic P_{n-2}-torsion A_{n-2}-module and $L'_{n-1} = H_1(\partial_n; M_{n-1})$ is zero or some copies of E_{n-2}.

Note that $M_{n-2} = H_0(\partial_{n-1}, \partial_n; M_n) = H_0(\partial_n; M_{n-1})$. By 2.1 we have $H_1(\partial_{n-1}; E_{n-1}) = 0$ and $H_0(\partial_{n-1}; E_{n-1}) = E_{n-2}$. It follows that $H_2(\partial_{n-1}, \partial_n; M_n) = 0$. Set $L_{n-2} = H_1(\partial_{n-1}, \partial_n; M_n)$. By 1.2 we have an exact sequence
$$0 \to H_0(\partial_{n-1}; L_{n-1}) \to L_{n-2} \to L'_{n-1} \to 0.$$
It follows that L_{n-2} is zero or some copies of E_{n-2}.

We iterate this process to obtain (after relabeling) $M_2 = H_0(\partial_3, \ldots, \partial_n; M_n)$ a graded holonomic $A_2(K)$-module. Also $L_2 = H_1(\partial_3, \ldots, \partial_n; M_n)$ is zero or some copies of E_2. Furthermore $H_i(\partial_3, \ldots, \partial_n; M_n) = 0$ for $i \geq 2$. Since P_2 is a height 0 prime in R_2, it is zero. So our process ends.

Clearly $H_i(\partial_1, \partial_2; M_2) = 0$ for $i \geq 3$. By 2.3, $H_i(\partial_1, \partial_2; L_2) = 0$ for $i \geq 1$. It follows that $H_i(\partial_1, \ldots, \partial_n; M_n)$ is zero for $i \geq 3$. $\square$

We now give

Proof of Theorem 4. $H^{n-2}_P(R)$ is a graded holonomic $A_n(K)$-module. It is also P-torsion. The result follows from Theorem 9.12. $\square$

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